

THE SECOND MIXED PROJECTION PROBLEM AND THE PROJECTION CENTROID CONJECTURES

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ABSTRACT. We prove the following four results: 1) There is a $C^2(S^{n-1})$ -neighborhood of the unit ball that the only convex bodies solving $\Pi_i^2 K = cK + \vec{v}$, for some $1 < i < n-1$, are balls. 2) There is a $C^2(S^{n-1})$ -neighborhood of the unit ball that the only convex bodies solving $\Gamma \Pi^* K = cK + \vec{v}$ are ellipsoids. 3) There is a $C^2(S^{n-1})$ -neighborhood of the unit ball that the only convex bodies solving $\Gamma \Pi_i^* K = cK + \vec{v}$, for some $1 \leq i < n-1$, are balls. 4) There is a $C(S^{n-1})$ -neighborhood of the unit ball that the only convex bodies solving $(\Pi \Gamma K)^* = cK$ are origin-centered ellipsoids. These results provide partial answers to open problems 4.5, 4.6 of [4] and 12.9 of [17].

1. INTRODUCTION

The setting of this paper is n -dimensional Euclidean space \mathbb{R}^n . A compact convex subset of \mathbb{R}^n with non-empty interior is called a *convex body*. The set of convex bodies in \mathbb{R}^n is denoted by K^n . Write K_e^n for the set of origin-symmetric convex bodies. Also, write B^n and S^{n-1} for the unit ball and the unit sphere of \mathbb{R}^n . Moreover, ω_k denotes the volume of B^k .

The support function of $K \in K^n$, $h_K : S^{n-1} \rightarrow \mathbb{R}$, is defined by

$$h_K(u) = \max_{x \in K} x \cdot u.$$

Assume $K \in K^n$, $n \geq 2$. The i th projection body $\Pi_i K$ of K is the origin-symmetric convex body whose support function, for $u \in S^{n-1}$, is given by

$$h_{\Pi_i K}(u) = \frac{1}{2} \int_{S^{n-1}} |u \cdot x| dS_i(K, x),$$

where $S_i(K, \cdot)$ is the mixed area measure of i copies of K and $n-1-i$ copies of B^n ; see [4, Section A3]. Note that Π_{n-1} coincides with the usual projection operator Π . We refer the reader to [16], especially Proposition 2, regarding the importance of classification of solutions to $\Pi_i^2 K = cK + \vec{v}$, where c is a positive constant. Let us remark that $\Pi_i B^n = \omega_{n-1} B^n$ and $\Pi_i^2 B^n = \omega_{n-1}^2 B^n$. [4, Problems 4.6] and [17, Problems 12.7] ask which convex bodies K are such that $\Pi_i^2 K$ is homothetic to K . The case $i = n-1$ has received partial answers; see [10, 19, 23]. Schneider [20] deals with the case $i = 1$ and proves origin-centered balls are the only solutions to $\Pi_1^2 K = cK$. Grinberg and Zhang [5] provide an alternative path to this result. Motivated by the work of Fish, Nazarov, Ryabogin and Zvavitch [3] where the idea of considering the iteration problems locally was first considered, here we prove

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local uniqueness theorems for fixed points of the second mixed projection operators for $1 < i < n - 1$:

Theorem 1.1. *Suppose $n \geq 3$ and $1 < i < n - 1$. There exists $\varepsilon > 0$ with the following property. If a convex body K satisfies $\Pi_i^2 K = cK + \vec{v}$ for some $c > 0$ and $\|h_{\lambda K + \vec{a}} - 1\|_{C^2} \leq \varepsilon$ for some $\lambda > 0$, then K is a ball.*

A set K in \mathbb{R}^n is called star-shaped if it is non-empty and if $[0, x] \subset K$ for every $x \in K$. For a compact star-shaped set K , the radial function ρ_K is defined by

$$\rho_K(x) = \max\{\lambda \geq 0 : \lambda x \in K\}, \quad x \in \mathbb{R}^n - \{0\}.$$

A compact star-shaped set with a positive continuous radial function is called a star body.

The polar body, K^* , of a convex body K with the origin in its interior is the convex body defined by

$$K^* = \{x \in \mathbb{R}^n : x \cdot y \leq 1 \text{ for all } y \in K\}.$$

It follows from the definition that $\rho_{K^*} = \frac{1}{\rho_K}$ on S^{n-1} .

The centroid body of a star body K is an origin-symmetric convex body whose support function, for $u \in S^{n-1}$, is given by

$$h_{\Gamma K}(u) = \int_{S^{n-1}} |u \cdot x| \rho_K^{n+1}(x) dx.$$

By a result of Petty, the centroid body of a convex body is always of class C_+^2 and $\Gamma\phi K = \phi\Gamma K$ for all $\phi \in \text{Sl}_n$; see [12] and [4, Theorem 9.1.3]. For $K \in \mathcal{K}_e^n$, ΓK is the locus of centroids of halves of K formed by slicing K by hyperplanes through the origin.

The curvature image of $K \in \mathcal{K}_e^n$, ΛK , is the origin-symmetric convex body whose positive, continuous curvature function (for the definition of the curvature function see [21, page 545]) $f_{\Lambda K}$ is given by

$$f_{\Lambda K} = \frac{1}{h_K^{n+1}}.$$

Moreover, $\Lambda\phi K = \phi\Lambda K$ for $\phi \in \text{Sl}_n$; see [14, Lemma 7.12].

Two conjectures of Lutwak stated in [17, Problem 12.9] are as follows: 1) If the convex body K is such that K and $\Gamma(\Pi K)^*$ are dilates, must K be an ellipsoid?¹ 2) If the star body K is such that K and $(\Pi\Gamma K)^*$ are dilates, does it follow that K is an ellipsoid? By Petty's regularity theorem for centroid bodies, if $\Gamma(\Pi K)^*$ and K are dilates then K is origin-symmetric and of class C_+^2 . Also, by a result of Martinez-Maure [18], the projection body of a convex body of class C_+^2 is C_+^2 . Thus, if K is a star body such that $(\Pi\Gamma K)^*$ and K are dilates, then K must be an origin-symmetric convex body of class C_+^2 . Furthermore, if $K \in \mathcal{K}_e^2$, then $\Pi K = 2K^{\frac{\pi}{2}}$ (rotation of K counter-clockwise through 90°). Therefore, $\Pi\Pi^* K = \Gamma\frac{1}{2}(K^{\frac{\pi}{2}})^* = \frac{1}{4}\Pi\Lambda K^{\frac{\pi}{2}} = \frac{1}{4}\Pi(\Lambda K)^{\frac{\pi}{2}} = \frac{1}{2}\Lambda K$. Consequently, if $\Pi\Pi^* K = cK$ for some positive constant c , then $\Lambda K = 2cK$. By a result of Petty [11, Lemma 8.1], K is an origin-centered ellipse. Similarly, if $K \in \mathcal{K}_e^2$ and $(\Pi\Gamma K)^*$ and K are dilates, then K is an origin-centered ellipse. In conclusion, the answer to both questions in \mathbb{R}^2 is positive. For higher dimensions, we prove the following results:

¹From now on, for simplicity, we set $(\Pi_i K)^* = \Pi_i^* K$.

Theorem 1.2. *Suppose $n \geq 3$. There exists $\varepsilon > 0$ with the following properties.*

- (1) *Let K be a convex body K such that $(\Gamma\Pi^*)^2K = cK + \vec{v}$ for some $c > 0$ and $\|h_{\phi K + \vec{a}} - 1\|_{C^2} \leq \varepsilon$ for some $\phi \in \text{Gl}_n$, then K is an ellipsoid.*
- (2) *Suppose $1 \leq i < n - 1$. If a convex body K satisfies $(\Gamma\Pi_i^*)^2K = cK + \vec{v}$ for some $c > 0$ and $\|h_{\lambda K + \vec{a}} - 1\|_{C^2} \leq \varepsilon$ for some $\lambda > 0$, then K is a ball.*
- (3) *If a star body K satisfies $(\Pi\Pi^*)^*K = cK$ for some $c > 0$ and $\|\rho_{\phi K} - 1\| \leq \varepsilon$ for some $\phi \in \text{Gl}_n$, then K is an origin-centered ellipsoid.*

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2. PRELIMINARIES

A convex body is said to be of class C_+^2 if its boundary hypersurface is two times continuously differentiable, in the sense of differential geometry, and has everywhere positive Gauss-Kronecker curvature.

Let S_k be the group of all the permutations of the set $\{1, \dots, k\}$. The mixed discriminant of functions $f_i \in C^2(S^{n-1})$, $1 \leq i \leq n - 1$, is a multi-linear operator defined as

$$(2.1) \quad \mathcal{Q}(f_1, \dots, f_{n-1}) := \frac{1}{(n-1)!} \sum_{\delta, \tau \in S_{n-1}} (-1)^{\text{sgn}(\delta) + \text{sgn}(\tau)} \prod_{i=1}^{n-1} (A[f_i])_{\delta(i)\tau(i)},$$

where in a local orthonormal frame of S^{n-1} the entries of the matrix $A[f_k]$ are given by $(A[f_k])_{ij} = \nabla_i \nabla_j f_k + \delta_{ij} f_k$ and ∇ is the covariant derivative on S^{n-1} . From the above definition it follows that the operator \mathcal{Q} is independent of the order of its arguments; see [1, Lemma 2-12] for other important properties of \mathcal{Q} .

We define the (ordered) mixed volume of $f_i \in C^2(S^{n-1})$, $1 \leq i \leq n$, by

$$V(f_1, \dots, f_n) := \frac{1}{n} \int_{S^{n-1}} f_1 \mathcal{Q}(f_2, \dots, f_n) dx.$$

In general we may have $V(f_1, \dots, f_n) \neq V(f_{\delta(1)}, \dots, f_{\delta(n)})$ for $\delta \in S_n$, but if $A[f_k]$ are all positive definite then the equality holds; see [1, Lemma 2-12].

The mixed projection of $f_i \in C^2(S^{n-1})$, $1 \leq i \leq n - 1$, is defined as

$$\Pi(f_1, \dots, f_{n-1})(u) = \frac{1}{2} \int_{S^{n-1}} |u \cdot x| \mathcal{Q}(f_1, \dots, f_{n-1})(x) dx.$$

Remark 2.1. Let $\{h_{K_i}\}_{1 \leq i \leq n}$ be the support functions of convex bodies $\{K_i\}$ of class C_+^2 . Then $V(h_{K_1}, \dots, h_{K_n})$ agrees with the usual definition of the mixed volume of K_1, \dots, K_n and also $\Pi(h_{K_1}, \dots, h_{K_{n-1}}) = h_{\Pi(K_1, \dots, K_{n-1})}$.

For convenience we will put

$$\begin{aligned}
\mathcal{Q}(f, \dots, f) &= \mathcal{Q}(f), \quad \mathcal{Q}(\underbrace{f, \dots, f}_{i \text{ times}}, g, \dots, g) = \mathcal{Q}_i(f, g), \\
\mathcal{Q}_i(f, 1) &= \mathcal{Q}_i(f), \quad \mathcal{Q}(\underbrace{f, \dots, f}_{i-1 \text{ times}}, g, \underbrace{1, \dots, 1}_{n-1-i \text{ times}}) = q_i(f, g), \\
V(f, \dots, f) &= V(f), \quad V(\underbrace{f, \dots, f}_{i \text{ times}}, g, \dots, g) = V_i(f, g), \quad V_i(f, 1) = V_i(f), \\
\Pi(f, \dots, f) &= \Pi f, \quad \Pi(\underbrace{f, \dots, f}_{i \text{ times}}, g, \dots, g) = \Pi_i(f, g), \quad \Pi_i(f, 1) = \Pi_i f.
\end{aligned}$$

2.1. Spherical harmonics. Write $L^2(S^{n-1})$ for the Hilbert space of square-integrable real functions on S^{n-1} equipped with scalar product

$$(f, g) := \int_{S^{n-1}} f g dx.$$

Write $\|\cdot\|_2$ for the induced norm by this scalar product.

Spherical harmonics of degree k are eigenfunctions of the spherical Laplace operator Δ with the eigenvalue $k(k+n-2)$. In fact, if Y_k is such function then

$$\Delta Y_k = -k(k+n-2)Y_k.$$

The set \mathcal{S}^k of spherical harmonics of degree k is a vector subspace of $C(S^{n-1})$. Moreover, $\dim \mathcal{S}^k = N(n, k) = \frac{2k+n-2}{k+n-2} \binom{k+n-2}{k}$. In each space \mathcal{S}^k , choose an orthonormal basis $\{Y_{k,1}, \dots, Y_{k,N(n,k)}\}$. For any $f \in L^2(S^{n-1})$ we write

$$\pi_k f := \sum_{l=1}^{N(n,k)} (f, Y_{k,l}) Y_{k,l}, \quad \pi_0 f = \frac{1}{n\omega_n} \int_{S^{n-1}} f dx.$$

The condensed harmonic expansion of f is given by

$$f \sim \sum_{k=0}^{\infty} \pi_k f;$$

it converges to f in the $L^2(S^{n-1})$ -norm. In addition, for $f, g \in L^2(S^{n-1})$ we have

$$\sum_{k=0}^{\infty} \sum_{l=1}^{N(n,k)} (f, Y_{k,l}) (g, Y_{k,l}) = (f, g).$$

Note that $f \in L^2(S^{n-1})$ if and only if its condensed harmonic expansion satisfies

$$\sum_{k=0}^{\infty} \|\pi_k f\|_2^2 < \infty.$$

For an excellent source on spherical harmonics see [8].

2.2. Radon transform and cosine transform. Suppose that f is a Borel function on S^{n-1} . The spherical Radon transform and cosine transform of f are defined as follows

$$\mathcal{R}f(u) = \frac{1}{(n-1)\omega_{n-1}} \int_{S^{n-1} \cap u^\perp} f(x) dx, \quad \mathcal{C}f(u) = \int_{S^{n-1}} |u \cdot x| f(x) dx.$$

The transformations \mathcal{R} and \mathcal{C} are self-adjoint, in the sense that if f and g are bounded Borel functions on S^{n-1} , then

$$\int_{S^{n-1}} f(x) \mathcal{R}g(x) dx = \int_{S^{n-1}} g(x) \mathcal{R}f(x) dx, \quad \int_{S^{n-1}} f(x) \mathcal{C}g(x) dx = \int_{S^{n-1}} g(x) \mathcal{C}f(x) dx.$$

Radon transform and cosine transform of a spherical harmonic of degree k are given by

$$\mathcal{R}Y_k = v_{k,n} Y_k, \quad v_{k,n} = (-1)^{\frac{k}{2}} \cdot \begin{cases} \frac{1 \cdot 3 \cdots (k-1)}{(n-1)(n+1) \cdots (n+k-3)}, & k \geq 4 \text{ even}; \\ 1, & k = 0; \\ \frac{1}{n-1}, & k = 2; \\ 0, & k \text{ odd} \end{cases}$$

and

$$\mathcal{C}Y_k = w_{k,n} Y_k, \quad w_{k,n} = (-1)^{\frac{k-2}{2}} \omega_{n-1} \cdot \begin{cases} 2 \frac{1 \cdot 3 \cdots (k-3)}{(n+1)(n+3) \cdots (n+k-1)}, & k \geq 4 \text{ even}; \\ 2, & k = 0; \\ \frac{2}{n+1}, & k = 2; \\ 0, & k \text{ odd}, \end{cases}$$

see [8, Lemma 3.4.5, 3.4.7]. The following relation between Radon transform and $\square := \Delta + n - 1$ is established in [7, Proposition 2.1]:

$$(2.2) \quad \square \mathcal{C} = 2(n-1)\omega_{n-1} \mathcal{R}.$$

We will also work with the spaces $H^s(S^{n-1})$, $s \geq 0$, of those functions for which the spherical harmonic expansion satisfies $\|f\|_{H^s}^2 = \sum_{k=0}^{\infty} (1+k^2)^s \|\pi_k f\|_2^2 < \infty$. These are precisely the functions $f \in L^2(S^{n-1})$ with derivatives derivations up to order s in $L^2(S^{n-1})$. Therefore, $H^s(S^{n-1})$, $s \geq 0$, are Sobolev spaces on S^{n-1} .

The following results about the smoothing property of \mathcal{R} , \mathcal{C} are proved in [22]:

$$(2.3) \quad \|\mathcal{R}f\|_{H^{s+\frac{n-2}{2}}} \leq a_{s,n} \|f\|_{H^s}, \quad \|\mathcal{C}f\|_{H^{s+\frac{n+2}{2}}} \leq b_{s,n} \|f\|_{H^s}$$

for some positive constants depending on s and n .

2.3. The directional derivative. Let B_1 and B_2 be two Banach spaces, U an open subset of B_1 . Suppose $P : U \subset B_1 \rightarrow B_2$ is continuous. The directional derivative of P at $f \in U$ in direction $g \in B_1$ is defined by

$$DP\{f, g\} = \lim_{t \rightarrow 0} \frac{P(f + tg) - P(f)}{t}.$$

If the limit exists, P is said to be differentiable at f in direction g . We say P is C^1 in U , if the limit exists for all $f \in U$ and $h \in B_1$ and if $DP : (U \subset B_1) \times B_1 \rightarrow B_2$ is continuous. The second derivative of P is the derivative of the first derivative:

$$D^2P\{f, g_1, g_2\} = \lim_{t \rightarrow 0} \frac{DP\{f + tg_2, g_1\} - DP\{f, g_1\}}{t}.$$

We say P is C^2 if D^2P exists and $D^2P : (U \subset B_1) \times B_1 \times B_1 \rightarrow B_2$ is jointly continuous on the product. Similar definitions apply to the higher derivatives. The k th derivative $D^kP\{f, g_1, \dots, g_k\}$ will be regarded as a map

$$D^kP : (U \subset B_1) \times \underbrace{B_1 \times \dots \times B_1}_{k \text{ times}} \rightarrow B_2.$$

We say P is of class C^k , if D^kP exists and is continuous. We say a map is C^∞ if it is C^k for all k .

If $P : U \subset B_1 \rightarrow V \subset B_2$ is a map between open subsets of Banach spaces, we define its tangent map $TP : (U \subset B_1) \times B_1 \rightarrow (V \subset B_2) \times B_2$ by

$$TP(f, g) = (P(f), DP\{f, g\}).$$

Note that TP is defined and continuous if and only if P is C^1 . Let $T^kP = T(T^{k-1}P)$, then T^kP is defined and continuous if and only if P is C^k . If P and Q are C^k , then their composition is also C^k and $T^k(P \circ Q) = T^kP \circ T^kQ$; see [9, Theorem 3.6.4].

3. THE MIXED PROJECTION PROBLEM

If $f \in C(S^{n-1})$, it follows from [18, Theorem 1.1] that $\mathcal{C}f \in C^2(S^{n-1})$. In particular, if $f \in C^2(S^{n-1})$, then $\Pi_i^k f \in C^2(S^{n-1})$.

Lemma 3.1. *The operator $\Pi_i^k : C^2(S^{n-1}) \rightarrow C^2(S^{n-1})$ is C^∞ .*

Proof. Since $\Pi_i = \frac{1}{2}\mathcal{C} \circ \mathcal{Q}_i$ and $\mathcal{C} : C(S^{n-1}) \rightarrow C^2(S^{n-1})$ is linear, it suffices to show that the operator $\mathcal{Q}_i : C^2(S^{n-1}) \rightarrow C(S^{n-1})$ is C^m for all m . Since \mathcal{Q} is multi-linear, we have

$$\mathcal{Q}_i(f + tg_1) - \mathcal{Q}_i(f) = it\mathcal{Q}_i(\underbrace{f, \dots, f}_{i-1 \text{ times}}, g_1, 1, \dots, 1) + o(t^2).$$

Therefore,

$$(3.1) \quad D\mathcal{Q}_i\{f, g_1\} = i\mathcal{Q}_i(\underbrace{f, \dots, f}_{i-1 \text{ times}}, g_1, 1, \dots, 1) = iq_i(f, g_1).$$

Consequently,

$$D\mathcal{Q}_i\{f + tg_2, g_1\} - D\mathcal{Q}_i\{f, g_1\} = i(i-1)t\mathcal{Q}_i(\underbrace{f, \dots, f}_{i-2 \text{ times}}, g_1, g_2, 1, \dots, 1) + o(t^2).$$

By induction we obtain

$$D^m\mathcal{Q}_i\{f, g_1, \dots, g_m\} = \begin{cases} \frac{i!}{(i-m)!}\mathcal{Q}_i(\underbrace{f, \dots, f}_{i-m \text{ times}}, g_1, g_2, \dots, g_m, 1, \dots, 1) & m \leq i; \\ 0 & m > i. \end{cases}$$

This explicit expression shows that $D^m\mathcal{Q}_i$ is defined and continuous. \square

Suppose $f \in C^2(S^{n-1})$ is the support function of a convex body of class C_+^2 . Let \tilde{U}_f be a $C^2(S^{n-1})$ -neighborhood of 0 such that for every $g \in \tilde{U}_f$, $f + g$ is the support function of a convex body of class C_+^2 . The Corollary on page 13 of [18] implies that $\Pi_i^k(f + g)$, for any k , is the support function of a C_+^2 convex body.

Lemma 3.2. Suppose $h \in C^2(S^{n-1})$ is the support function of a convex body of class C_+^2 . For $g \in C^2(S^{n-1})$ we have

$$\left. \frac{d}{dt} \right|_{t=0} \Pi_i^k(h + tg) = \frac{i^k}{2^k} \mathcal{C}q_i(\Pi_i^{k-1}h, \mathcal{C}q_i(\Pi_i^{k-2}h, \mathcal{C}q_i(\dots, \mathcal{C}q_i(h, g) \dots)))$$

and

$$\left. \frac{d}{dt} \right|_{t=0} \Pi_i^k(1 + tg) = \frac{\left(\frac{V(\Pi^k 1)}{\omega_n} \right)^{\frac{1}{n}} i^k}{2^k \omega_{n-1}^k (n-1)^k} (\mathcal{C}\square)^{(k)} g,$$

where $(\mathcal{C}\square)^{(k)} g = \underbrace{\mathcal{C}\square \dots \mathcal{C}\square}_k g$. Furthermore,

$$\begin{aligned} & \left. \frac{d}{dt} \right|_{t=0} V_{i+1}(\Pi_i^k(h + tg)) \\ &= \frac{i^k(i+1)}{2^{k-1}n} \int_{S^{n-1}} gq_i(h, \mathcal{C}q_i(\Pi_i h, \mathcal{C}q_i(\dots, \mathcal{C}q_i(\Pi_i^{k-1}h, \Pi_i^{k+1}h) \dots))) dx \end{aligned}$$

and

$$\left. \frac{d}{dt} \right|_{t=0} V_{i+1}(\Pi_i^k(1 + tg)) = \frac{\left(\frac{V(\Pi^k 1)}{\omega_n} \right)^{\frac{1}{n}} i^k(i+1)}{2^k \omega_{n-1}^k (n-1)^k n} \left((\square \mathcal{C})^{(k)} (\Pi^k 1)^i \right) \int_{S^{n-1}} g dx.$$

Proof. Note that $T\Pi_i^k = \underbrace{T\Pi_i \circ \dots \circ T\Pi_i}_k$ and by (3.1) we have

$$T\Pi_i(h, g) = (\Pi_i h, \frac{i}{2} \mathcal{C}q_i(h, g)).$$

Thus

$$T\Pi_i^2(h, g) = T\Pi_i(\Pi_i h, \frac{i}{2} \mathcal{C}q_i(h, g)) = (\Pi_i^2 h, \frac{i}{2} \mathcal{C}q_i(\Pi_i h, \frac{i}{2} \mathcal{C}q_i(h, g))).$$

The general claim follows by induction.

Note for a fixed g , there exists $\varepsilon > 0$ small enough, such that for any $t \in (-\varepsilon, \varepsilon)$, $tg \in \tilde{U}_h$; therefore, $A[h + tg]$ is positive definite and it is the support function of a convex body of class C_+^2 . To calculate $\left. \frac{d}{dt} \right|_{t=0} V_{i+1}(\Pi_i^k(h + tg))$, we may restrict our attention only to the range of $t \in (-\varepsilon, \varepsilon)$ (although, the definition of the derivative implicitly considers only small t and so such care was not needed). Recall that

$$V_{i+1}(\Pi_i^k(h + tg)) = \frac{1}{n} \int_{S^{n-1}} \Pi_i^k(h + tg) \mathcal{Q}_i(\Pi_i^k(h + tg)) dx.$$

Therefore,

$$\begin{aligned}
\frac{d}{dt}\Big|_{t=0} V_{i+1}(\Pi_i^k(h+tg)) &= \frac{1}{n} \int_{S^{n-1}} \mathcal{Q}_i(\Pi_i^k h) \frac{d}{dt}\Big|_{t=0} \Pi_i^k(h+tg) dx \\
&\quad + \frac{1}{n} \int_{S^{n-1}} \Pi_i^k h \frac{d}{dt}\Big|_{t=0} \mathcal{Q}_i(\Pi_i^k(h+tg)) dx \\
&= \frac{1}{n} \int_{S^{n-1}} \mathcal{Q}_i(\Pi_i^k h) \frac{d}{dt}\Big|_{t=0} \Pi_i^k(h+tg) dx \\
&\quad + \frac{i}{n} \int_{S^{n-1}} \Pi_i^k h q_i(\Pi_i^k h, \frac{d}{dt}\Big|_{t=0} \Pi_i^k(h+tg)) dx.
\end{aligned}$$

Using [1, Lemma 2-12, items (3),(4)], we get

$$\frac{d}{dt}\Big|_{t=0} V_{i+1}(\Pi_i^k(h+tg)) = \frac{i+1}{n} \int_{S^{n-1}} \mathcal{Q}_i(\Pi_i^k h) \frac{d}{dt}\Big|_{t=0} \Pi_i^k(h+tg) dx.$$

Since the operator \mathcal{C} is self-adjoint and $\mathcal{C}\mathcal{Q}_i(\Pi_i^k h) = 2\Pi_i^{k+1}h$, in view of [1, Lemma 2-12, items (3),(4)] the claim follows. For the special case of $h = 1$, we refer the reader to the proofs of [10, Lemmas 3.2, 3.4]. \square

Fix $i, m \in \mathbb{N}$. Suppose $f \in C^2(S^{n-1})$ is the support function of a convex body of class C_+^2 . Define a map by

$$\begin{aligned}
\mathcal{X}_{i,f}^m : \tilde{U}_f \subset C^2(S^{n-1}) &\rightarrow C^2(S^{n-1}) \\
\mathcal{X}_{i,f}^m(g)(u) &:= -\Pi_i^m(f+g)(u) + \left(\frac{V_{i+1}(\Pi_i^m(f+g))}{V_{i+1}(f+g)} \right)^{\frac{1}{1+i}} (f+g)(u) \\
&\quad - u \cdot \int_{S^{n-1}} \left(\frac{V_{i+1}(\Pi_i^m(f+g))}{V_{i+1}(f+g)} \right)^{\frac{1}{1+i}} (f+g)(x) x dx.
\end{aligned}$$

By Lemmas 3.1 and 3.2, $\mathcal{X}_{i,f}^m$ is C^∞ . Lemma 3.2 yields an explicit expression for $D\mathcal{X}_{i,1}^{2m}\{0, \cdot\}$:

Lemma 3.3. *For any $g \in C^2(S^{n-1})$ we have*

$$\begin{aligned}
D\mathcal{X}_{i,1}^{2m}\{0, g\}(u) &= (\Pi^{2m}1) \left(g(u) - i^{2m} \mathcal{R}^{2m}g(u) + \frac{i^{2m}-1}{n\omega_n} \int_{S^{n-1}} g dx \right) \\
&\quad - (\Pi^{2m}1)u \cdot \int_{S^{n-1}} g(x) x dx.
\end{aligned}$$

Furthermore, if $i < n-1$, then $\dim \text{Ker } D\mathcal{X}_{i,1}^{2m}\{0, \cdot\} = n+1$.

Proof. Using Lemma 3.2, we calculate

$$\begin{aligned}
 D\mathcal{X}_{i,1}^{2m}\{0, g\}(u) &= \left(-\frac{\left(\frac{V(\Pi^{2m}1)}{\omega_n}\right)^{\frac{1}{n}} i^{2m}}{2^{2m}\omega_{n-1}^{2m}(n-1)^{2m}} (\mathcal{C}\square)^{(2m)} g + \left(\frac{V_{i+1}(\Pi^{2m}1)}{\omega_n}\right)^{\frac{1}{i+1}} g \right. \\
 &\quad + \left(\frac{V_{i+1}(\Pi^{2m}1)}{\omega_n}\right)^{\frac{1}{i+1}-1} \frac{\left(\frac{V(\Pi^{2m}1)}{\omega_n}\right)^{\frac{1}{n}} i^{2m}}{2^{2m}\omega_{n-1}^{2m}(n-1)^{2m}} \left((\square\mathcal{C})^{(2m)} (\Pi^{2m}1)^i \right) \frac{\int_{S^{n-1}} g dx}{n\omega_n} \\
 &\quad - \left(\frac{V_{i+1}(\Pi^{2m}1)}{\omega_n}\right)^{\frac{1}{i+1}} \frac{\int_{S^{n-1}} g dx}{n\omega_n} \Bigg) (u) \\
 &\quad - u \cdot \left\{ \int_{S^{n-1}} \left(\left(\frac{V_{i+1}(\Pi^{2m}1)}{\omega_n}\right)^{\frac{1}{i+1}} g \right. \right. \\
 &\quad + \left(\frac{V_{i+1}(\Pi^{2m}1)}{\omega_n}\right)^{\frac{1}{i+1}-1} \frac{\left(\frac{V(\Pi^{2m}1)}{\omega_n}\right)^{\frac{1}{n}} i^{2m}}{2^{2m}\omega_{n-1}^{2m}(n-1)^{2m}} \left((\square\mathcal{C})^{(2m)} (\Pi^{2m}1)^i \right) \frac{\int_{S^{n-1}} g dx}{n\omega_n} \\
 &\quad \left. \left. - \left(\frac{V_{i+1}(\Pi^{2m}1)}{\omega_n}\right)^{\frac{1}{i+1}} \frac{\int_{S^{n-1}} g dx}{n\omega_n} \right) (x) dx \right\}.
 \end{aligned}$$

On the other hand,

$$(\Pi^{2m}1)^i = \left(\frac{V_{i+1}(\Pi^{2m}1)}{\omega_n}\right)^{-\frac{1}{i+1}+1}, \quad \left(\frac{V(\Pi^{2m}1)}{\omega_n}\right)^{\frac{1}{n}} = \left(\frac{V_{i+1}(\Pi^{2m}1)}{\omega_n}\right)^{\frac{1}{i+1}}.$$

Substituting these back into the above identity completes the proof.

To find the $\dim \text{Ker } D\mathcal{X}_{i,1}^{2m}\{0, \cdot\}$, note that

$$(3.2) \quad u \cdot \int_{S^{n-1}} g(x) x dx = \pi_1 g(u);$$

see [21, p. 50]. Therefore,

$$D\mathcal{X}_{i,1}^{2m}\{0, g\} = \begin{cases} 0, & g \in \mathcal{S}_0 \oplus \mathcal{S}_1; \\ (\Pi^{2m}1)(1 - i^{2m}v_{k,n}^{2m})\pi_k g, & g \in \mathcal{S}_k \quad k \geq 2. \end{cases}$$

Thus $\dim \text{Ker } D\mathcal{X}_{i,1}^{2m}\{0, \cdot\} = n + 1$. \square

Lemma 3.4. *Suppose $m \geq 4$ and $1 < i < n - 1$. Given $h \in C^2(S^{n-1})$ with $\pi_k h = 0$ for $k = 0, 1$, there exists a unique $g \in C^2(S^{n-1})$ with $\pi_k g = 0$ for $k = 0, 1$ such that*

$$g - i^{2m}\mathcal{R}^{2m}g = h.$$

Proof. We develop h into a series of spherical harmonics: $h \sim \sum_{k \geq 2}^{\infty} \pi_k h$. Since $L^2(S^{n-1})$ is a complete space and $\lim_{k \rightarrow \infty} 1 - i^{2m}v_{k,n}^{2m} = 1$, the $L^2(S^{n-1})$ -Cauchy sequence

$$\left\{ f_l := \sum_{k \geq 2}^l \frac{1}{1 - i^{2m}v_{k,n}^{2m}} \pi_k h \right\}_l$$

converges in the $L^2(S^{n-1})$ -norm to a bounded $f \in L^2(S^{n-1}) \cap (\mathcal{S}^0 \oplus \mathcal{S}^1)^\perp$ with

$$\pi_k f = \frac{1}{1 - i^{2m} v_{k,n}^{2m}} \pi_k h$$

for $k \geq 2$. In view of (2.3), $\mathcal{R}^{2m} f \in H^{m(n-2)} \subset H^{4(n-2)} \subset C^2(S^{n-1})$. Define

$$g := h + i^{2m} \mathcal{R}^{2m} f.$$

Note that $g \in C^2(S^{n-1}) \cap (\mathcal{S}^0 \oplus \mathcal{S}^1)^\perp$ and for $k \geq 2$:

$$\pi_k g = \left(1 + \frac{i^{2m} v_{k,n}^{2m}}{1 - i^{2m} v_{k,n}^{2m}}\right) \pi_k h \Rightarrow \pi_k (g - i^{2m} \mathcal{R}^{2m} g) = \pi_k h.$$

Since h and $g - i^{2m} \mathcal{R}^{2m} g$ are C^2 , we conclude that

$$(3.3) \quad g - i^{2m} \mathcal{R}^{2m} g = h$$

The uniqueness claim: Suppose $g_1, g_2 \in C^2(S^{n-1})$ both solve (3.3) with $\pi_k g_i = 0$ for $k = 0, 1$. Therefore, for $k \geq 2$,

$$(1 - i^{2m} v_{k,n}^{2m}) \pi_k g_i = \pi_k h \Rightarrow \pi_k g_i = \frac{\pi_k h}{1 - i^{2m} v_{k,n}^{2m}} \Rightarrow \pi_k g_1 = \pi_k g_2 \Rightarrow g_1 = g_2.$$

□

Theorem 3.5. *Suppose $m \geq 4$ and $1 < i < n - 1$. There exists $\varepsilon_m > 0$, such that if K satisfies $\Pi_i^{2m} K = cK + \vec{v}$ for some $c > 0$ and $\|h_{\lambda K + \vec{a}} - 1\|_{C^2} \leq \varepsilon_m$ for some $\lambda > 0$, then K is a ball. In particular, if $\Pi_i^2 K = cK + \vec{v}$ for some $c > 0$ and $\|h_{\lambda K + \vec{a}} - 1\|_{C^2} \leq \varepsilon_4$ for some $\lambda > 0$, then K is a ball.*

Proof. Fix $1 \leq i < n - 1$ and $m \geq 4$. In this proof, B always denotes a ball. Consider the map

$$\mathcal{N} : \tilde{U}_1 \subset C^2(S^{n-1}) \rightarrow C^2(S^{n-1}) \quad f \mapsto \mathcal{X}_{i,1}^{2m}(f) + (\pi_0 + \pi_1)(f), \quad \mathcal{N}(0) = 0.$$

- \mathcal{N} is C^∞ .
- By Lemma 3.4, $D\mathcal{N}\{0, \cdot\}$ is an invertible linear map.²

By the inverse function theorem (see, [9, Theorem 5.2.3, Corollary 5.3.4]), we can find neighborhoods U, W of 0 in $C^2(S^{n-1})$, such that $\mathcal{N} : U \subset \tilde{U}_1 \rightarrow W$ is a smooth diffeomorphism. Put $M := \mathcal{N}^{-1}(W \cap \text{Ker } D\mathcal{X}_{i,1}^{2m}\{0, \cdot\})$. Observe that

$$f \in U \text{ and } \mathcal{X}_{i,1}^{2m}(f) = 0 \Rightarrow \mathcal{N}^{-1}((\pi_0 + \pi_1)(f)) = f \in M.$$

On the other hand, if $h_B - 1 \in U$, then $\mathcal{X}_{i,1}^{2m}(h_B - 1) = 0$ and clearly

$$W' := \{(\pi_0 + \pi_1)(h_B - 1); h_B - 1 \in U\} = \{h_B - 1; h_B - 1 \in U\}$$

forms an open subset of $W \cap \text{Ker } D\mathcal{X}_{i,1}^{2m}\{0, \cdot\}$ about the origin in \mathbb{R}^{n+1} .

Define the open set

$$W_1 := \{f \in W; (\pi_0 + \pi_1)f \in W'\}.$$

Thus $W' = W_1 \cap \text{Ker } D\mathcal{X}_{i,1}^{2m}\{0, \cdot\}$ and $\mathcal{N}^{-1}(W') = M \cap \mathcal{N}^{-1}(W_1)$. Since $\mathcal{N}^{-1}(W_1)$ is an open neighborhood of 0 in $C^2(S^{n-1})$ and $\mathcal{N}^{-1}(W') = \{h_B - 1; h_B - 1 \in U\}$,

²Given $h \in C^2(S^{n-1})$, by Lemma 3.4 there exists $g \in C^2(S^{n-1})$ such that

$$g - i^{2m} \mathcal{R}^{2m} g = h - (\pi_0 + \pi_1)h$$

and $\pi_k g = 0$ for $k = 0, 1$. Define $l = g + (\pi_0 + \pi_1)h$. Then $l \in C^2(S^{n-1})$ and $D\mathcal{N}(0, l) = h$.

we conclude that in a C^2 -neighborhood of 1, the only solutions of $\mathcal{X}_{i,1}^{2m}(\cdot - 1) = 0$ are balls.

So far we have shown that there exists $\varepsilon_m > 0$, such that if K satisfies $\mathcal{X}_{i,1}^{2m}(h_K - 1) = 0$ and $\|h_K - 1\|_{C^2} \leq \varepsilon_m$, then K is a ball. To see that the first claim of the theorem holds, note that if $\Pi_i^{2m}K = cK + \vec{v}$, then $\mathcal{X}_{i,1}^{2m}(h_{\lambda K + \vec{a}} - 1) = 0$; therefore, K is a ball.

To prove the second statement of the theorem, note that $\Pi_i^2K = cK + \vec{v}$ yields $\mathcal{X}_{i,1}^8(h_{\lambda K + \vec{a}} - 1) = 0$; therefore, K is a ball. \square

4. THE PROJECTION CENTROID CONJECTURES

For a convex body K of class C_+^2 , define

$$\Theta_i(h_K) := h_{\Gamma \Pi_i^* K} = \mathcal{C}((\Pi_i h_K)^{-(n+1)}) = 2^{n+1} \mathcal{C} \frac{1}{(\mathcal{C} \mathcal{Q}_i(h_K))^{n+1}}.$$

Recall that $\mathcal{C} \mathcal{Q}_i(h_K) = 2\Pi_i h_K$ is the support function of an origin-symmetric convex body of class C_+^2 and consequently $2/\mathcal{C} \mathcal{Q}_i(h_K)$ is the radial function (restricted to S^{n-1}) of the corresponding polar body (which is also an origin-symmetric convex body of class C_+^2). Petty's regularity theorem for centroid bodies ensures $\Theta_i(h_K)$ is the support function of a convex body of class C_+^2 . By induction, for any $m \in \mathbb{N}$,

$$\Theta_i^m(h_K) := \underbrace{\Theta_i \circ \cdots \circ \Theta_i}_{m \text{ times}}(h_K)$$

is the support function of a convex body of class C_+^2 .

Define the map

$$\begin{aligned} \mathcal{Y}_{i,1}^m : \tilde{U}_1 \subset C^2(S^{n-1}) &\rightarrow C^2(S^{n-1}) \\ \mathcal{Y}_{i,1}^m(f)(u) &= \left(-\Theta_i^m(1+f) + \left(\frac{V(\Theta_i^m(1+f))}{V(1+f)} \right)^{\frac{1}{n}} (1+f) \right) (u) \\ &\quad - u \cdot \int_{S^{n-1}} \left(\frac{V(\Theta_i^m(1+f))}{V(1+f)} \right)^{\frac{1}{n}} (1+f)(x) x dx. \end{aligned}$$

Since $\Pi_i : C^2(S^{n-1}) \rightarrow C^2(S^{n-1})$ is C^∞ , by the chain rule the map $\mathcal{Y}_{i,1}^m$ is C^∞ . Furthermore, we calculate

$$\begin{aligned} T\Theta_i(h_K, g) &= 2^{n+1} T\mathcal{C} \circ T \frac{1}{x^{n+1}} \circ T\mathcal{C} \circ T\mathcal{Q}_i(h_K, g) \\ &= 2^{n+1} T\mathcal{C} \circ T \frac{1}{x^{n+1}} \circ T\mathcal{C}(\mathcal{Q}_i(h_K), i q_i(h_K, g)) \\ &= 2^{n+1} T\mathcal{C} \circ T \frac{1}{x^{n+1}} (\mathcal{C} \mathcal{Q}_i(h_K), i \mathcal{C} q_i(h_K, g)) \\ &= 2^{n+1} T\mathcal{C} \left(\frac{1}{(\mathcal{C} \mathcal{Q}_i(h_K))^{n+1}}, -i(n+1) \frac{\mathcal{C} q_i(h_K, g)}{(\mathcal{C} \mathcal{Q}_i(h_K))^{n+2}} \right) \\ &= (\Theta_i(h_K), -i(n+1) 2^{n+1} \mathcal{C} \frac{\mathcal{C} q_i(h_K, g)}{(\mathcal{C} \mathcal{Q}_i(h_K))^{n+2}}) \end{aligned}$$

and

$$\begin{aligned}
T\Theta_i^2(h_K, g) &= T\Theta_i \circ T\Theta_i(h_K, g) \\
&= T\Theta_i(\Theta_i(h_K), -i(n+1)2^{n+1}\mathcal{C} \frac{\mathcal{C}q_i(h_K, g)}{(\mathcal{C}\mathcal{Q}_i(h_K))^{n+2}}) \\
(4.1) \quad &= (\Theta_i^2(h_K), -i(n+1)2^{n+1}\mathcal{C} \frac{\mathcal{C}q_i(\Theta_i(h_K), -i(n+1)2^{n+1}\mathcal{C} \frac{\mathcal{C}q_i(h_K, g)}{(\mathcal{C}\mathcal{Q}_i(h_K))^{n+2}})}{(\mathcal{C}\mathcal{Q}_i(\Theta_i(h_K)))^{n+2}}).
\end{aligned}$$

Lemma 4.1. *For any $g \in C^2(S^{n-1})$ we have*

$$\begin{aligned}
D\mathcal{Y}_{i,1}^2\{0, g\}(u) &= \Theta_i^2(1) \left(g(u) - \frac{i^2(n+1)^2}{4\omega_{n-1}^2} \mathcal{C}^2 \mathcal{R}^2 g(u) + \frac{i^2(n+1)^2 - 1}{n\omega_n} \int_{S^{n-1}} g dx \right) \\
&\quad - \Theta_i^2(1) u \cdot \int_{S^{n-1}} g(x) x dx.
\end{aligned}$$

Also, We have

$$\dim \text{Ker } D\mathcal{Y}_{i,1}^2\{0, \cdot\} = \begin{cases} \frac{n(n+3)}{2}, & i = n-1; \\ n+1, & 1 \leq i < n-1. \end{cases}$$

Proof. Note that $\Theta_i(h_{\lambda K}) = \frac{1}{\lambda^{i(n+1)}} \Theta_i(h_K)$. Hence we get

$$\Theta_i^2(1) = \Theta_i(\Theta_i(1)) = \frac{1}{(\Theta_i(1))^{i(n+1)-1}}.$$

In view of $\Theta_i(1) = 2/\omega_{n-1}^n$ and the identity (4.1) for $h_K \equiv 1$ we obtain

$$\begin{aligned}
\frac{1}{\Theta_i^2(1)} \frac{d}{dt} \Big|_{t=0} \Theta_i^2(1 + tg) &= \frac{1}{\Theta_i^2(1)} \frac{i^2 2^{2n+4} (n+1)^2 (\Theta_i(1))^{i-1} \omega_{n-1}^2}{(\Theta_i(1))^{i(n+2)} (2\omega_{n-1})^{2n+4}} \mathcal{C}^2 \mathcal{R}^2 g \\
&= \frac{i^2(n+1)^2}{(\Theta_i(1))^2 (\omega_{n-1})^{2n+2}} \mathcal{C}^2 \mathcal{R}^2 g \\
(4.2) \quad &= \frac{i^2(n+1)^2}{4\omega_{n-1}^2} \mathcal{C}^2 \mathcal{R}^2 g.
\end{aligned}$$

Using this last expression, we calculate

$$\begin{aligned}
\frac{1}{\Theta_i^2(1)} \frac{d}{dt} \Big|_{t=0} V(\Theta_i^2(1 + tg)) &= \int_{S^{n-1}} \frac{i^2(n+1)^2}{4\omega_{n-1}^2} \mathcal{C}^2 \mathcal{R}^2 g \mathcal{Q}(\Theta_i^2(1)) dx \\
&= (\Theta_i^2(1))^{n-1} i^2(n+1)^2 \int_{S^{n-1}} g dx.
\end{aligned}$$

Thus

$$(4.3) \quad \frac{1}{\Theta_i^2(1)} \frac{d}{dt} \Big|_{t=0} \left(\frac{V(\Theta_i^2(1 + tg))}{V(1 + tg)} \right)^{\frac{1}{n}} = \frac{i^2(n+1)^2 - 1}{n\omega_n} \int_{S^{n-1}} g dx.$$

Putting (4.2), (4.3) together yields the explicit expression for $D\mathcal{Y}_{i,1}^2\{0, g\}$. In view of (3.2), it is easy to check that $D\mathcal{Y}_{i,1}^2\{0, g\} = 0$ for all $g \in \mathcal{S}_0 \oplus \mathcal{S}_1$. Also, from $\mathcal{C}g = \frac{2\omega_{n-1}}{n+1}g$ and $\mathcal{R}g = -\frac{1}{n-1}g$ for all $g \in \mathcal{S}_2$, it follows that $\mathcal{S}_0 \oplus \mathcal{S}_1 \oplus \mathcal{S}_2 \subset$

$\text{Ker } D\mathcal{Y}_{n-1,1}^2\{0, \cdot\}$. To complete the proof, note that if $k \geq 3$ and $i = n-1$ or $k \geq 2$ and $i < n-1$, then $1 - \frac{i^2(n+1)^2}{4\omega_{n-1}^2}v_{k,n}^2w_{k,n}^2 \neq 0$. \square

Lemma 4.2. *The following statements hold.*

- (1) *Given $h \in C^2(S^{n-1})$ with $\pi_k h = 0$ for $k = 0, 1, 2$, there exists a unique $g \in C^2(S^{n-1})$ with $\pi_k g = 0$ for $k = 0, 1, 2$ such that*

$$g - \frac{(n-1)^2(n+1)^2}{4\omega_{n-1}^2}\mathcal{C}^2\mathcal{R}^2g = h.$$

- (2) *Given $h \in C^2(S^{n-1})$ with $\pi_k h = 0$ for $k = 0, 1$, there exists a unique $g \in C^2(S^{n-1})$ with $\pi_k g = 0$ for $k = 0, 1$ such that*

$$g - \frac{i^2(n+1)^2}{4\omega_{n-1}^2}\mathcal{C}^2\mathcal{R}^2g = h.$$

Proof. We only give the proof of the first claim. Develop h into a series of spherical harmonics: $h \sim \sum_{k \geq 3}^\infty \pi_k h$. The $L^2(S^{n-1})$ -Cauchy sequence

$$\left\{ f_l := \sum_{k \geq 3}^l \frac{1}{1 - \frac{(n-1)^2(n+1)^2}{4\omega_{n-1}^2}v_{k,n}^2w_{k,n}^2} \pi_k h \right\}_l$$

converges in the $L^2(S^{n-1})$ -norm to a bounded $f \in L^2(S^{n-1}) \cap (\mathcal{S}^0 \oplus \mathcal{S}^1 \oplus \mathcal{S}^2)^\perp$ with

$$\pi_k f = \frac{1}{1 - \frac{(n-1)^2(n+1)^2}{4\omega_{n-1}^2}v_{k,n}^2w_{k,n}^2} \pi_k h$$

for $k \geq 3$. In view of (2.3), $\mathcal{C}^2\mathcal{R}^2 f \in H^{2n} \subset C^2(S^{n-1})$. Set

$$g := h + \frac{(n-1)^2(n+1)^2}{4\omega_{n-1}^2}\mathcal{C}^2\mathcal{R}^2 f.$$

We have $g \in C^2(S^{n-1}) \cap (\mathcal{S}^0 \oplus \mathcal{S}^1 \oplus \mathcal{S}^2)^\perp$ and for $k \geq 3$

$$\pi_k g = \left(1 + \frac{\frac{(n-1)^2(n+1)^2}{4\omega_{n-1}^2}v_{k,n}^2w_{k,n}^2}{1 - \frac{(n-1)^2(n+1)^2}{4\omega_{n-1}^2}v_{k,n}^2w_{k,n}^2} \right) \pi_k h.$$

Therefore,

$$\pi_k \left(g - \frac{(n-1)^2(n+1)^2}{4\omega_{n-1}^2}\mathcal{C}^2\mathcal{R}^2g \right) = \pi_k h.$$

Since $h, g - \frac{(n-1)^2(n+1)^2}{4\omega_{n-1}^2}\mathcal{C}^2\mathcal{R}^2g \in C^2(S^{n-1})$, we obtain

$$g - \frac{(n-1)^2(n+1)^2}{4\omega_{n-1}^2}\mathcal{C}^2\mathcal{R}^2g = h.$$

The uniqueness claim is trivial. \square

Given Lemmas 4.1 and 4.2, proofs of the first and second statements in Theorem 1.2 are straightforward; if $1 \leq i < n-1$, we precisely follow the proof of Theorem 1.1 and for the case $i = n-1$, we refer the reader to the proof of [10, Theorem 4.2].

All constants c_i that follow are positive. We proceed to the proof of the third statement. Note that for any $L \in K_e^n$:

$$\Gamma L = 2\Pi\Lambda L^*, \quad \Gamma\psi K = \psi\Gamma K \text{ for any } \psi \in \text{Sl}_n.$$

We may assume without loss of generality that $\phi \in \text{Sl}_n$. We have

$$(\Pi\Gamma\phi K)^* = c_1\phi K \Rightarrow \Pi\phi\Gamma K = \frac{1}{c_1}(\phi K)^*.$$

Therefore,

$$\Pi\Lambda\Pi\phi\Gamma K = c_2\Pi\Lambda(\phi K)^* = \frac{c_2}{2}\Gamma\phi K \Rightarrow \Gamma\Pi^*(\phi\Gamma K) = c_3\phi\Gamma K.$$

On the other hand, if δ is small enough, then from $\|\rho_{\phi K} - 1\| \leq \delta$ it follows that $\|\rho_{\phi K}^{n+1} - 1\| \leq 2\delta$. Thus by [10, Ineq. (2.3)],

$$\|h_{\frac{1}{2\omega_{n-1}}}\phi\Gamma K - 1\|_{C^2} = \frac{1}{2\omega_{n-1}}\|\mathcal{C}\rho_{\phi K}^{n+1} - \mathcal{C}1\|_{C^2} \leq c_n\delta.$$

In summary, we have shown that $\Gamma\Pi^*(\frac{1}{2\omega_{n-1}}\phi\Gamma K) = \frac{c_4}{2\omega_{n-1}}\phi\Gamma K$ and $h_{\frac{1}{2\omega_{n-1}}}\phi\Gamma K$ satisfies the assumption (1) in Theorem 1.2 provided δ is small enough. Hence $\phi\Gamma K$ is an origin-centered ellipsoid. Since $\Pi\phi\Gamma K = \frac{1}{c_1}(\phi K)^*$, K is an origin-centered ellipsoid.

REFERENCES

1. B. Andrews, “Entropy inequalities for evolving hypersurfaces,” *Comm. Anal. Geom.* 2(1994), 53–64.
2. T. Bonnesen, W. Fenchel, “*Theorie der konvexen Körper*,” Springer-Verlag, Berlin, 1934.
3. A. Fish, F. Nazarov, D. Ryabogin, A. Zvavitch, “The unit ball is an attractor of the intersection body operator,” *Adv. in Math.* 226(2011), 2629–2642.
4. R. Gardner, “*Geometric Tomography*,” Vol. 6. Cambridge: Cambridge University Press, 2006.
5. E. Grinberg and G. Zhang, “Convolutions, transforms, and convex bodies,” *Proc. London Math. Soc.* 78(1999), 77–115.
6. P.R. Goodey, H. Groemer, “Stability results for first order projection bodies,” *Proc. Amer. Math. Soc.* 109(1990), 1103–1114.
7. P.R. Goodey, W. Weil, “Centrally symmetric convex bodies and the spherical Radon transform,” *J. Differ. Geom.* 5(1992), 675–688.
8. H. Groemer, “*Geometric applications of Fourier series and spherical harmonics*,” Vol. 61. Cambridge University Press, 1996.
9. R.S. Hamilton, “The inverse function theorem of Nash and Moser,” *Bull. Amer. Math. Soc. (N.S.)* 7(1982), 65–222.
10. M.N. Ivaki, “A local uniqueness theorem for minimizers of Petty’s conjectured projection inequality,” preprint arXiv:1610.03796v1, (2016).
11. C.M. Petty, “Affine isoperimetric problems,” *Annals of the New York Academy of Sciences* 440(1985), 113–127.
12. C.M. Petty, “Centroid surfaces,” *Pacific J. Math.* 11(1961), 1535–1547.
13. E. Lutwak, “On some affine isoperimetric inequalities,” *J. Differential Geom.* 23(1986), 1–13.
14. E. Lutwak, “Centroid bodies and dual mixed volumes,” *Proc. London Math. Soc.* 60(1990), 365–391.
15. E. Lutwak, “On a conjectured projection inequality of Petty,” *Contemp. Math.* 113(1990), 171–182.
16. E. Lutwak, “On quermassintegrals of mixed projection bodies,” *Geom. Dedicata* 33(1990), 51–58.
17. E. Lutwak, “Selected affine isoperimetric inequalities,” In *Handbook of Convex Geometry*, ed. by P.M. Gruber and J.M. Wills. North-Holland, Amsterdam, 1993, pp. 151–176.
18. Y. Martinez-Maure, “Hedgehogs and zonoids,” *Adv. in Math.* 158(2001), 1–17.

19. C. Saroglou, A. Zvavitch, "Iterations of the projection body operator and a remark on Petty's conjectured projection inequality," J. Funct. Anal. (2016), DOI:10.1016/j.jfa.2016.08.015.
20. R. Schneider, "Rekonstruktion eines konvexen Körpers aus seinen Projektionen," Math. Nachr. 79(1977), 325-329
21. R. Schneider, "Convex bodies: the Brunn–Minkowski theory," No. 151. Cambridge University Press, 2013.
22. R.S. Strichartz, " L^p estimates for Radon transforms in Euclidean and non-Euclidean spaces," Duke Math. J. 48(1981), 699–727.
23. W. Weil, "Über die Projektionskörper konvexer Polytope," Arch. Math. 22(1971): 664–672.

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